

A Theory of Non-Gaussian Option Pricing

Lisa Borland
Evnine-Vaughan Associates, Inc.
456 Montgomery Street, Suite 800
San Francisco, CA 94104-1241, USA

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Abstract

Option pricing formulas are derived from a non-Gaussian model of stock returns. Fluctuations are assumed to evolve according to a nonlinear Fokker-Planck equation which maximizes the Tsallis nonextensive entropy of index q . A generalized form of the Black-Scholes differential equation is found, and we derive a martingale measure which leads to closed form solutions for European call options. The standard Black-Scholes pricing equations are recovered as a special case ($q = 1$). The distribution of stock returns is well-modelled with q circa 1.5. Using that value of q in the option pricing model we reproduce the volatility smile. The partial derivatives (or Greeks) of the model are also calculated. Empirical results are demonstrated for options on Japanese Yen futures. Using just one value of σ across strikes we closely reproduce market prices, for expiration times ranging from weeks to several months.

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1 Introduction

It is well known that the distributions of empirical returns do not follow the lognormal distribution upon which many celebrated results of finance are based. For example, Black and Scholes [1] and Merton [2] were able to derive the prices of options and other derivatives of the underlying stock based on such a model. While of great importance and widely used, such theoretical option prices do not quite match the observed ones. In particular, the Black-Scholes model underestimates the prices of away-from-the-money options. This means that the implied volatilities of options of various strike prices form a convex function, rather than the expected flat line. This is known as the “volatility smile”.

Indeed, there have been several modifications to the standard models in an attempt to correct for these discrepancies. One approach is to introduce a stochastic model for the volatility of the stock price, as was done by Hull and White [3], or via a generalized autoregressive conditional heteroskedasticity (GARCH) model of volatility. A review and references can be found in [5]. Another class of models include a Poisson jump diffusion term [4] which can describe extreme price movements. The DVF (Deterministic Volatility Function) approach [6], as well as combinations of some of these different approaches [7], have also been studied. A quite different line of thought is offered in [8, 9], where it is argued that heavy non-Gaussian tails and finite hedging time make it necessary to go beyond the notion of risk-free option prices. They obtain non-unique prices, associated with a given level of risk. More recently, other techniques along the lines of [10] lead to option prices based on an underlying hyperbolic distribution.

In many cases, these approaches are often either very complicated or rather ad-hoc. To our knowledge, none result in manageable closed form solutions, which is a useful result of the Black and Scholes approach. In this paper we do however succeed in obtaining closed form solutions for European options. Our approach is based on a new class of stochastic processes which allow for statistical feedback as a model of the underlying stock returns. We can show that the distributions of returns implied by these processes closely match those found empirically. In particular they capture features such as the fat tails and peaked middles which are not at all captured by the standard class of lognormal distributions.

Our stochastic model derives from a class of processes [11] which have

been recently developed within the framework of statistical physics, namely within the very active field of Tsallis nonextensive thermostatics [12]. Many interesting applications of this new statistical paradigm have been found in recent years, mainly related to the sciences, although there are some results showing that the power-law distributions characteristic of the Tsallis framework are good models for the distributions of certain financial quantities [13, 14, 15]. However, to our knowledge the current work, a short version of which is given in [16], contains the first application of the associated stochastic processes to finance.

Basically, these stochastic processes can be interpreted as if the driving noise follows a generalized Wiener process governed by a fat-tailed Tsallis distribution of index $q > 1$. For $q = 1$ the Tsallis distribution coincides with a Gaussian and the standard stock-price model is recovered. However, for $q > 1$ these distributions exhibit fat tails and appear to be good models of real data, as shown in Figure 1. There, the empirical distribution of the log daily price returns (ignoring dividends and non-trading days) to the demeaned S&P 500 is plotted. Returns were normalized by the sample standard deviation of the series which is 19.86 % annualized, and then binned. For comparison, the distribution obtained from a Tsallis distribution of index $q = 1.43$ is also plotted [15]. It seems clear that the Tsallis distribution provides a much better fit to the empirical distribution than the lognormal, which is also shown. Another example is shown in Figure 2, where the distribution of high frequency log returns for 10 Nasdaq high-volume stocks is plotted [15]. The timescale is 1 minute. Again, returns are normalized by the sample standard deviation. A Tsallis distribution of index $q = 1.43$ provides a very good fit to the empirical data. Another example of such a match between Tsallis distributions ($q = 1.6$) and those of financial returns over different timescales can be found in [14].

Motivated by the good fit between the proposed model class and empirical data, we use these stochastic processes to represent movements of the returns of the underlying stock. We then derive generalized option pricing formulas so as to be able to obtain fair values of derivatives of the underlying. Using these formulas we get a good match with empirically observed option prices. In particular, we show in this paper (see Figures 16 and 17) that a $q = 1.4$ model with one value of σ across strikes reproduces market prices for options on Japanese Yen futures with expiry dates ranging from 17 to 147 days.

2 The Model of Returns

The standard model for stock price movement is that

$$S(\tau + t) = S(\tau)e^{Y(t)} \quad (1)$$

where Y follows the stochastic process

$$dY = \mu dt + \sigma d\omega \quad (2)$$

The drift μ is the mean rate of return and σ^2 is the variance of the stock logarithmic return. The driving noise ω is a Brownian motion defined with respect to a probability measure F . It represents a Wiener process and has the property

$$E^F[d\omega(t)d\omega(t')] = dt dt' \delta(t - t') \quad (3)$$

where the notation $E^F[\cdot]$ means the expectation value with respect to the measure F . Note that the conditional probability distribution of the variable ω satisfies the Fokker-Planck equation

$$\frac{\partial P(\omega, t \mid \omega', t')}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial \omega^2} P(\omega, t \mid \omega', t') \quad (4)$$

and is distributed according to

$$P(\omega(t), t \mid \omega(t'), t') = \frac{1}{\sqrt{2\pi(t - t')}} \exp\left(-\frac{(\omega(t) - \omega(t'))^2}{2(t - t')}\right) \quad (5)$$

In addition one chooses $t' = 0$ and $\omega(0) = 0$ so that this defines a Wiener process, which is distributed according to a zero-mean Gaussian.

It is well-known that this model gives a normal distribution with drift μt and variance $\sigma^2 t$ for the variable Y . This can for example be seen by rewriting Eq (2) as

$$d\left(\frac{Y - \mu t}{\sigma}\right) = d\omega \quad (6)$$

which indicates that we can substitute

$$\omega = (Y - \mu t)/\sigma \quad (7)$$

into Eq (5). We obtain the well-known lognormal distribution for the stock returns over timescale T , after inserting $Y = \ln S(\tau + t)/S(\tau)$:

$$P(\ln S(T + \tau) | \ln S(\tau)) = N \exp\left(-\frac{(\ln \frac{S(\tau+T)}{S(\tau)} - \mu T)^2}{2\sigma^2 T}\right) \quad (8)$$

Based on this stock-price model, Black and Scholes were able to establish a pricing model to obtain the fair value of options on the underlying stock S . However, their model predicts a lognormal distribution, whereas empirical distributions of stock returns are better fitted with power-law distributions [9, 17, 14, 15]. Here we focus mainly on the empirical evidence of [14, 15] where it is shown that the distributions which naturally arise within the framework of the generalized thermostatics of Tsallis [12] provide very good fits to empirical distributions of returns on different timescales.

In contrast to other models where the standard Black-Scholes price model is extended to account for non-normal noise, for example jump diffusion models [4] and Levy noise [10], we introduce here a new model of stock return fluctuations, which derives directly from stochastic processes recently introduced within the Tsallis framework [11]. In this setting, we assume that the log returns $Y(t) = \ln S(\tau + t)/\ln S(\tau)$ follow the process

$$dY = \mu dt + \sigma d\Omega \quad (9)$$

across timescales t , where we shall now model the driving noise Ω as being drawn from a non-Gaussian distribution. To do this, we assume that Ω follows the statistical feedback process [11]

$$d\Omega = P(\Omega)^{\frac{1-q}{2}} d\omega \quad (10)$$

Here ω is a zero-mean Gaussian noise process as defined above. For $q = 1$, Ω reduces to ω and the standard model is recovered. The probability distribution of the variable Ω evolves according to the nonlinear Fokker-Planck equation [11]

$$\frac{\partial}{\partial t} P(\Omega, t | \Omega', t') = \frac{1}{2} \frac{\partial}{\partial \Omega^2} P^{2-q}(\Omega, t | \Omega', t') \quad (11)$$

It can be verified that the conditional probability P that solves this system is given by so-called Tsallis distributions (or q-Gaussians P_q)

$$P_q(\Omega, t | \Omega', t') = \frac{1}{Z(t)} \left(1 - \beta(t)(1 - q)(\Omega - \Omega')^2\right)^{\frac{1}{1-q}} \quad (12)$$

with

$$\beta(t) = c^{\frac{1-q}{3-q}} ((2-q)(3-q)(t-t'))^{-2/(3-q)} \quad (13)$$

and

$$Z(t) = ((2-q)(3-q)c(t-t'))^{\frac{1}{3-q}} \quad (14)$$

By choosing $t' = 0$ and $\Omega(0) = 0$, we obtain a generalized Wiener process, distributed according to a zero-mean Tsallis distribution

$$P_q(\Omega, t | 0, 0) = \frac{1}{Z(t)} \left(1 - \beta(t)(1-q)(\Omega)^2\right)^{\frac{1}{1-q}} \quad (15)$$

The index q is known as the entropic index of the generalized Tsallis entropy. The q -dependent constant c is given by

$$c = \beta Z^2 \quad (16)$$

$$Z = \int_{-\infty}^{\infty} (1 - (1-q)\beta\Omega^2)^{\frac{1}{1-q}} d\Omega \quad (17)$$

for any β . In the limit $q \rightarrow 1$ the standard theory is recovered, and P_q becomes a Gaussian. In that case, the standard Gaussian driving noise of Eq (2) is also recovered. For $q < 1$ these distributions exhibit a so-called cutoff resulting in regimes where $P_q = 0$. In the current paper, we will therefore only consider values of $q > 1$, for which the distributions exhibit fat tails. There is also a natural limit at $q = 3$ after which value the distributions are no longer normalizable. Another important point which constrains the realistic range of q -values is the fact that the variance of the Tsallis distributions is given by [19]

$$E[\Omega^2(t)] = \frac{1}{(5-3q)\beta(t)} \quad (18)$$

Clearly, this expression diverges for $q \geq 5/3$. Since we are only interested in processes with finite variance, we assume $1 < q < 5/3$, which covers the values of empirical interest.

Our model exhibits a statistical feedback into the system, from the macroscopic level characterised by P , to the microscopic level characterised by the dynamics of Ω , and thereby ultimately by the returns Y . This scenario is simply a phenomenological description of the underlying dynamics. For example, in the case of stock prices, we can imagine that the statistical feedback is really due to the interactions of many individual traders whose actions all

will contribute to shocks to the stock price which keep it in equilibrium. Their collective behaviour can be summarized by the statistical dependency in the noise term of the stochastic model for Y . This yields a nonhomogeneous reaction to the returns: depending on the value of q , rare events (i.e. extreme returns) will be accompanied by large reactions. On the other hand, if the returns take on less extreme values, then the size of the noise is more moderate.

The time-dependent solutions presented here can be seen as a special case of those presented in [18] where the δ -function as initial condition was not explicitly discussed. It is not difficult to verify that the particular form of P_q which we introduce here has the property that P_q becomes sharply peaked as t approaches zero. In other words, it approaches a δ -function as $t \rightarrow 0$, which corresponds to the fact that the returns are known with certainty to be zero over intervals $t = 0$.

Let us look at what effect the driving noise Ω has on the log returns $Y(t)$. We can write

$$d\left(\frac{Y - \mu t}{\sigma}\right) = d\Omega \quad (19)$$

or equivalently

$$\Omega(t) = (\ln \frac{S(\tau + t)}{S(\tau)} - \mu t)/\sigma \quad (20)$$

It then follows from Eq (12) that the distribution of returns $\ln S(\tau + t)/S(\tau)$ obeys

$$P_q(\ln S(\tau + t) | \ln S(\tau)) = \frac{1}{Z(t)} \left(1 - \tilde{\beta}(t)(1 - q)(\ln \frac{S(\tau + t)}{S(\tau)} - \mu t)^2 \right)^{\frac{1}{1-q}} \quad (21)$$

with $\tilde{\beta} = \beta(t)/\sigma^2$.

This implies that the distribution of log-returns $\ln S(\tau + t)/S(\tau)$ over the interval t follows a Tsallis distribution, evolving anomalously across timescales. This result is consistent with empirical evidence, in particular results found in [14] for the S & P 500. Consequently, the way in which the stochastic equation Eq(9) with Eq(10) should be interpreted is that it generates members $Y(t)$ of an ensemble of returns, distributed on each timescale t according to a non-Gaussian Tsallis distribution of index q . With such an interpretation, the current model should be applicable to pricing both standard and exotic options, except for such options which are explicitly dependent on

the history of evolution of a particular price path. However, in the current paper we only look at standard options. Exotics will be the topic of future study.

At each timescale the distribution is of Tsallis form of index q . This appears to be true empirically, based on studies in [14, 15], although for large t , empirical distributions do seem to become Gaussian. However, this is consistent with our model because as t increases, the central region of the Tsallis distribution is very well approximated by a Gaussian. Seeing as empirical data becomes sparser and sparser for large t , it is virtually impossible to say whether real returns become more and more Gaussian as the timescale increases, or whether they are still of Tsallis index q and only appear Gaussian due to a lack of empirical measurement in the tail region [15].

Another point which should be addressed is the fact that the noise distribution at each timescale evolves according to a Tsallis distribution, with variance scaling anomalously with timescale. These results are consistent with empirical observations on short timescales [14] but it is more commonly found that the variance of returns scales normally at larger timescales. Such scaling of the variance can be achieved with the current model simply by including a state dependent instantaneous rate of return, for example, if the log returns show some form of mean reversion. As shown in [11, 18], the nonlinear Fokker-Planck equation corresponding to such a problem yields as solutions Tsallis distributions of index q on each timescale, but the exponent of the temporal evolution of $\langle Y(t)^2 \rangle \propto 1/\beta(t)$ can be normal or even subdiffusive, depending on both q and the strength of the mean reverting term.

However, for the purpose of asset pricing using Martingale techniques, the explicit properties of the deterministic part of the dynamics of the log returns become irrelevant, as long as the Novikov condition Eq(45) is satisfied. We shall leave further exploration of these details for future work, but make a note that the results in this paper could probably be extended to be valid for certain classes of models with state-dependent μ .

3 Risk-Free portfolio and the Generalized Black-Scholes Differential Equation

Our model for log-returns reads

$$dY = \mu dt + \sigma d\Omega \quad (22)$$

with $Y(t) = \ln S(\tau + t)/S(\tau)$ and $d\Omega$ given by Eq(10). In the following we shall set $\tau = 0$ without loss of generality for our current discussion. The stock price itself follows

$$dS = \left(\mu + \frac{\sigma^2}{2} P_q^{1-q}\right) S dt + \sigma S d\Omega \quad (23)$$

which can be abbreviated as

$$dS = \tilde{\mu} S dt + \sigma S d\Omega \quad (24)$$

where

$$\tilde{\mu} = \mu + \frac{\sigma^2}{2} P_q^{1-q} \quad (25)$$

Remember that P_q (given by Eq(15)) is a function of $\Omega(t)$, so $\tilde{\mu}$ itself ultimately varies with time. (Having a time dependent rate of return is a perfectly valid assumption, even in the standard case). The term $\frac{\sigma^2}{2} P_q^{1-q}$ which appears here is none other than a noise-induced drift term. For $q = 1$ the standard noise-induced drift term is recovered. This stock-price model implies that log returns are distributed according to the Tsallis distribution of Eq(21). (Note that a fully equivalent treatment of the problem is to assume that the dynamics of the stock price is instead given by Eq(137), as discussed in Appendix A).

Let us now look at price movements of a derivative of the underlying stock S , modelled by Eq(24). We denote the price of the derivative by $f(S)$ and we use the stochastic (Ito) calculus to obtain

$$df = \frac{df}{dS} dS + \frac{df}{dt} dt + \frac{1}{2} \frac{d^2 f}{dS^2} (\sigma^2 P_q^{1-q}) dt \quad (26)$$

where in turn dS is given by Eq(24) with Eq(10). After insertion we get

$$df = \left(\frac{df}{dS} \tilde{\mu} S + \frac{df}{dt} + \frac{1}{2} \frac{d^2 f}{dS^2} (\sigma^2 S^2 P_q^{1-q}) \right) dt + \frac{df}{dS} \sigma S P_q^{\frac{1-q}{2}} d\omega \quad (27)$$

In the limit $q \rightarrow 1$, we recover the standard equations for price movements and derivatives thereof.

It is important to realize that the noise terms driving the price of the shares S is the same as that driving the price f of the derivative. It should be possible to invest one's wealth in a portfolio of shares and derivatives in such a way that the noise terms cancel each other, yielding the so-called risk-free portfolio. Following the same steps as in the standard case (cf [5][22]),

$$\Pi = -f + \frac{df}{dS}S \quad (28)$$

A small change in this portfolio is given by

$$\Delta\Pi = -\Delta f + \frac{df}{dS}\Delta S \quad (29)$$

which, after insertion of the expressions for f and S , becomes

$$\Delta\Pi = -\left(\frac{df}{dt} + \frac{1}{2}\frac{d^2f}{dS^2}\sigma^2S^2P_q^{1-q}\right)\Delta t \quad (30)$$

The return on this portfolio must be the risk-free rate r , otherwise there would be arbitrage opportunities. One thus gets the following generalized version of the Black-Scholes differential equation:

$$\frac{df}{dt} + \frac{1}{2}\frac{d^2f}{dS^2}\sigma^2S^2P_q^{1-q} = r\left(f - \frac{df}{dS}S\right) \quad (31)$$

or rather

$$\frac{df}{dt} + rS\frac{df}{dS} + \frac{1}{2}\frac{d^2f}{dS^2}\sigma^2S^2P_q^{1-q} = rf \quad (32)$$

where P_q evolves according to Eq(15). In the limit $q \rightarrow 1$, we recover the standard Black-Scholes differential equation.

This differential equation does not explicitly depend on μ , the rate of return of the stock, only on the risk-free rate and the variance. However, there is a dependency on $\Omega(t)$ through the term P_q . But it is possible to express $\Omega(t)$ in terms of $S(t)$ through Eq(20), which implies that there is an implicit dependency on μ . Therefore, to be consistent with risk-free pricing theory, we should first transform our original stochastic equation for S into a martingale before we apply the above analysis. This will not affect our results other than that $\tilde{\mu}$ will be replaced by the risk-free rate r . In the next Section we show how this can be done.

4 Equivalent Martingale Measures

Assume that there is a call option with strike price K written on the underlying asset $S(t)$. Its value will be given by

$$C(T) = \max[S(T) - K, 0] \quad (33)$$

at expiration date T . At earlier times $t < T$, the value of $C(T)$ is unknown but one can forecast it using the information $I(t)$ available up until time t , so

$$E^F[C(T) \mid I(t)] = E^F[\max[S(T) - K, 0] \mid I(t)] \quad (34)$$

where the notation $E^F[C]$ means that the expectation E of the random variable C is taken with respect to the probability measure F under which the dynamics of C (and thereby S) are defined. In addition, we must require that the fair market value $C(t)$, discounted accordingly in the risk-neutral framework at the risk-free rate, is equal to $E^F[\max[S(T) - K, 0] \mid I(t)]$. However, this is only true if $e^{-rt}S(t)$ satisfies the martingale condition

$$E^F[e^{-rt}S(t) \mid S(u), u < t] = e^{-ru}S(u) \quad (35)$$

This means that under the measure F , the conditional expectation of $S(t)$ discounted at the risk-free rate is best given by the discounted value of S at the previous time u . Heuristically one can say that a martingale is a stochastic process whose trajectories display no obvious trends or periodicities. A submartingale is a process that, on average, is increasing. For example, using the stock price model of Eq(24), we get for $G(t) = e^{-rt}S(t)$

$$dG = (\tilde{\mu} - r)Gdt + G\sigma d\omega \quad (36)$$

Clearly, G is a submartingale because of the non-zero drift term, whereas the process

$$dG - (\tilde{\mu} - r)Gdt = G\sigma d\omega \quad (37)$$

is a martingale. Subtracting the drift from a submartingale G in a somewhat similar manner is the basis of the so-called Doob-Meyer decomposition. If the drift term can be explicitly determined, then it is possible to decompose G into a drift component and a martingale component and thereby determine the fair market value of $C(t)$

However, this method is not usually used. Instead, it is common in asset-pricing to find synthetic probabilities Q under which the drift of the underlying stochastic process vanishes, i.e., find Q so that

$$E^Q[e^{-rt}S(t) \mid S(u), u < t] = e^{-ru}S(u) \quad (38)$$

In order to transform our probability-dependent stochastic processes into martingales, we will need to generalize several of the concepts used in the standard asset-pricing theory. Therefore, we shall first review the standard case.

If a stochastic process is given by

$$dY = \mu dt + \sigma d\omega \quad (39)$$

where ω is a Brownian noise term associated with a probability measure F , then it is not a martingale because of the drift term μdt . According to the Girsanov theorem, one can however find an equivalent measure Q corresponding to an alternative noise term dz , such that the process is transformed into a martingale, by rewriting it as

$$dY = \sigma \left(\frac{\mu}{\sigma} dt + d\omega \right) = \sigma dz \quad (40)$$

The new driving noise term z is related to ω through

$$z = \int_0^t u ds + \omega \quad (41)$$

with

$$u = \frac{\mu}{\sigma} \quad (42)$$

The noise term z is defined with respect to the equivalent Martingale measure Q which is related to F through the Radon-Nikodym derivative

$$\zeta(t) = \frac{dQ}{dF} = \exp \left(- \int_0^t u d\omega - \frac{1}{2} \int_0^t u^2 ds \right) \quad (43)$$

Under the measure F , the original random variable ω follows a zero-mean process with variance equal to t . Under that same measure, the new noise term $z(t)$ is normal with non-zero mean equal to $\int_0^t u ds$ and variance t . However, with respect to the equivalent probability measure Q one can

easily verify that $z(t)$ is normal with 0 mean and variance t . This follows because the relationship

$$E^Q[Y] = E^F[\zeta Y] \quad (44)$$

holds. In the above discussion, u , μ and σ may all depend on the variable $Y(t)$ as well. The only criterion which must be satisfied for the Girsanov theorem to be valid is that

$$\exp\left(-\frac{1}{2}\int_0^t u^2 ds\right) < \infty \quad (45)$$

which implies that ζ is a square integral martingale. This is known as the Novikov condition (for details see Oksendal [23]).

The effect of the martingale transformation is further illustrated by the conditional probability distribution of the variable Y : With respect to ω , P is given by

$$P(Y, t \mid Y(t_0), t_0) = \frac{1}{\sigma\sqrt{2\pi(t-t_0)}} \exp\left(\frac{-((Y - Y(t_0)) - \mu(t-t_0))^2}{\sigma^2(t-t_0)}\right) \quad (46)$$

which is Gaussian with drift $\mu(t-t_0)$. On the other hand, with respect to z , the probability distribution of Y is given by

$$P(Y, t \mid Y(t_0), t_0) = \frac{1}{\sigma\sqrt{2\pi(t-t_0)}} \exp\left(\frac{-(Y - Y(t_0))^2}{\sigma^2(t-t_0)}\right) \quad (47)$$

This is a Gaussian distribution with zero drift.

Now we would like to formulate similar equivalent martingale measures for the present class of probability dependent stochastic processes. Let the original process be given by

$$dY = \mu dt + \sigma d\Omega \quad (48)$$

with Ω defined as in Eq(10), namely

$$d\Omega = P_q^{\frac{1-q}{2}} d\omega \quad (49)$$

where ω is normally distributed δ -correlated noise, associated with the measure F . $P_q(\Omega)$ is the Tsallis distribution of index q discussed above Eq(15),

so with respect to the measure F , $P(Y, t | Y(t_0), t_0)$ is given by the non-zero drift distribution

$$P_q(Y, t | Y(t_0), t_0) = \frac{1}{Z(t)} \left(1 - \tilde{\beta}(t)(1-q)(Y - Y(t_0) - \mu(t - t_0))^2 \right)^{\frac{1}{1-q}} \quad (50)$$

We are now in the position to define equivalent Martingale measures exactly as in the standard case by writing

$$dY = \sigma P_q^{\frac{1-q}{2}} \left(\frac{\mu}{\sigma P_q^{\frac{1-q}{2}}} dt + d\omega \right) \quad (51)$$

$$= \sigma P_q^{\frac{1-q}{2}} dz \quad (52)$$

This new driving noise z is associated with the measure Q and reads

$$dz = \frac{\mu}{\sigma P_q^{\frac{1-q}{2}}} dt + d\omega \quad (53)$$

Let us define

$$u = \frac{\mu}{\sigma P_q^{\frac{1-q}{2}}} \quad (54)$$

Since P_q is simply a particular function of Ω , which in turn can be expressed as a function of Y via $\Omega = (Y - \mu t)/\sigma$, we are dealing with a general function $u(Y)$, so our analysis will be formally equivalent to that of the standard case. In particular, since P_q is a non-zero bounded function of Y the criterion Eq(45) is valid and thereby also the Girsanov theorem. The martingale equivalent measure Q under which z is defined is given by Eq (43) with u as in Eq(54). Under Q , the noise term z is a zero-mean Brownian motion. Remember that z and Q are merely synthetic measures. They are purely mathematical constructions that do not reflect the true probabilities or dynamics of Y .

The most important point that we shall utilize in this work is the following. Since z is under Q a zero-mean Gaussian noise, then the noise term defined by

$$d\Omega = P_q(\Omega)^{\frac{1-q}{2}} dz \quad (55)$$

is equivalent to that defined by Eq(10) and the distribution of the variable Ω is therefore given by a Tsallis distribution of index q . Consequently, under Q ,

the variable Y as defined by the stochastic equation Eq(52) is also distributed according to a zero-drift Tsallis distribution, namely

$$P_q(Y, t | Y(t_0), t_0) = \frac{1}{Z(t)} \left(1 - \tilde{\beta}(t)(1 - q)(Y - Y(t_0))^2 \right)^{\frac{1}{1-q}} \quad (56)$$

5 Transforming the Discounted Stock Price to a Martingale

In the following, we will discuss the problem of how to transform the discounted stock price into a martingale. Let the discounted stock price be

$$G = Se^{-rt} \quad (57)$$

such that

$$\ln G = \ln S - rt. \quad (58)$$

The model for S is given by Eq(24), yielding

$$dG = (\tilde{\mu} - r)Gdt + \sigma Gd\Omega \quad (59)$$

for the discounted stock price $G = Se^{-rt}$. The dynamics of Ω is defined with respect to the measure F as in Eq (10). Here, $\tilde{\mu}$ includes a noise-induced drift term and reads as in Eq(25). Stochastic integration shows that at time T we have

$$G(T) = G(0) \exp((\tilde{\mu} - r)T - \frac{\sigma^2}{2} \int_0^T P_q^{1-q} dt + \int_0^T \sigma P_q^{\frac{1-q}{2}} dz_s) \quad (60)$$

$$= G(0) \exp((\mu - r)T + \int_0^T \sigma P_q^{\frac{1-q}{2}} dz_s) \quad (61)$$

which implies

$$S(T) = S(0) \exp \left(\int_0^T \sigma P_q^{\frac{1-q}{2}} d\omega_s + \int_0^T (\tilde{\mu} - \frac{\sigma^2}{2} P_q^{1-q}) dt \right) \quad (62)$$

$$= S(0) \exp \left(\int_0^T \sigma P_q^{\frac{1-q}{2}} d\omega_s + \mu T \right) \quad (63)$$

These expressions are derived based on the original representation of the price dynamics, given by Eq(59). However it is clear that Eq(59) is not

a martingale, but can be transformed into one following the same ideas as discussed in the previous Section. We get

$$dG = \sigma G P_q^{\frac{1-q}{2}} dz \quad (64)$$

with

$$dz = \left(\frac{\tilde{\mu} - r}{\sigma P_q^{\frac{1-q}{2}}} \right) dt + d\omega \quad (65)$$

$$= \left(\frac{\mu - r + \frac{\sigma^2 P_q^{1-q}}{2}}{\sigma P_q^{\frac{1-q}{2}}} \right) dt + d\omega \quad (66)$$

$$(67)$$

Notice that P_q depends on Ω which in turn depends on S as was shown in Eq(20). S itself can be expressed in terms of G via Eq(57). Therefore, the rules of standard stochastic calculus can be applied, and the martingale equivalent measure Q associated with z is obtained from Eq(43) by setting

$$u = \frac{\mu - r + \frac{\sigma^2 P_q^{1-q}}{2}}{\sigma P_q^{\frac{1-q}{2}}} \quad (68)$$

Taking the log of Eq(64) we get

$$d \ln G = -\frac{\sigma^2}{2} P_q^{1-q} dt + \sigma P_q^{\frac{1-q}{2}} dz \quad (69)$$

After stochastic integration and transforming back to S we obtain

$$S(T) = S(0) \exp \left(\int_0^T \sigma P_q^{\frac{1-q}{2}} dz_s + \int_0^T \left(r - \frac{\sigma^2}{2} P_q^{1-q} \right) ds \right) \quad (70)$$

$$(71)$$

with

$$\alpha = \frac{1}{2}(3-q)((2-q)(3-q)c)^{\frac{q-1}{3-q}} \quad (72)$$

If we compare the expression Eq(70) for S under Q with that under F as given by Eq(62), we see that the difference between the two is that the rate

of return $\tilde{\mu}$ has been replaced by the risk-free rate r . This recovers the same result as in the standard risk-free asset pricing theory. (Exactly the same result Eq(70) would have been obtained had we instead started with the stock price model Eq(137), as mentioned in Appendix A. It is not hard to see that that would have been equivalent to substituting μ with the risk free rate r).

We have yet to discuss the evaluation of the P_q related terms which appear in the above expressions. Two points will be of importance here: The first being that the term of type $\int_0^T P_q^{\frac{1-q}{2}} dz$ is simply equal to the random variable $\Omega(T)$. The second important point (discussed in Appendix B) is to realize that the distributions $P_q(\Omega(s))$ at arbitrary times s can be mapped onto the distributions $P_q(\Omega(T))$ at a fixed time T via the appropriate variable transformation

$$\Omega(s) = \sqrt{\frac{\beta(T)}{\beta(s)}} \Omega(T) \quad (73)$$

Using these notions we can write $S(T)$ of Eq(70) as

$$\begin{aligned} S(T) &= S(0) \exp \left(\Omega(T) + rT - \frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} + (1-q) \frac{\sigma^2}{2} \int_0^T \frac{\beta(t)}{Z(t)^{1-q}} \Omega^2(t) dt \right) \\ &= S(0) \exp \left(\Omega(T) + rT - \frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} (1 - (1-q)\beta(T)\Omega^2(T)) \right) \end{aligned} \quad (74)$$

This expression for $S(T)$ recovers the usual one for $q = 1$. For $q > 1$, a major difference to the standard case is the $\Omega^2(T)$ -term in the exponential, which appears as a result of the noise induced drift. The implications of this term for the option prices will become apparent further on.

Let us revisit the generalized Black-Scholes PDE Eq(32). In the risk-neutral world, we must use Eq(69) to obtain an expression for $P_q(\Omega)$. Rewriting that equation yields

$$d \frac{(\ln S - rt + \frac{\sigma^2}{2} P_q^{1-q}(\Omega))}{\sigma} = d\Omega \quad (75)$$

Formally, this expression is identical to Eq(20) except that $\tilde{\mu}$ (related to μ through Eq(25)) has been replaced with r . Furthermore, integrating Eq(75) up to time t results in Eq(74) (with $T = t$), from which it is possible to solve for $\Omega(t)$ explicitly in terms of $S(t)$. This implies that, in the martingale

representation, $P_q(\Omega(t))$ can be expressed as a function of the volatility σ , the risk-free rate r , $S(t)$ and $S(0)$. Most importantly, the implicit dependency on μ through Ω is replaced by a dependency on r .

The generalized differential equation Eq(32) can thus be solved numerically, which is one way of obtaining option prices in this generalized framework. However, it is possible to go a step further and obtain closed-form option prices. This is done by transforming asset prices into martingales and then taking expectations. In the following Sections we show how this is done, and why the option prices obtained in that way indeed satisfy Eq(32).

6 The Generalized Option Pricing Formula

Suppose that we have a European claim C which depends on $S(t)$, whose price f is given by its expectation value in a risk-free (martingale) world as

$$f(C) = E^Q[e^{-rT}C] \quad (76)$$

If the payoff on this option depends on the stock price at the expiration time T so that

$$C = h(S(T)) \quad (77)$$

then we obtain

$$f = e^{-rT} E^Q \left[h \left(S(0) \exp \left(\int_0^T \sigma P_q^{\frac{1-q}{2}} dz_s + \int_0^T \left(r - \frac{\sigma^2}{2} P_q^{1-q} \right) ds \right) \right) \right] \quad (78)$$

In the special case of $q = 1$, the standard expression of the option price is recovered with this formula (see for example Oksendal [23]). However, in that case it is argued that under Q , the random variable

$$x(T) = \int_0^T \sigma dz \quad (79)$$

is normally distributed with variance

$$\delta^2 = \int_0^T \sigma^2 dt \quad (80)$$

yielding the following expression for a European claim:

$$f = \frac{e^{-rT}}{\delta\sqrt{2\pi}} \int_R h \left[S(0) \exp \left(x + \int_0^T \left(r - \frac{1}{2} \sigma^2(s) \right) ds \right) \right] \exp \left(-\frac{x^2}{2\delta^2} \right) dx \quad (81)$$

The key difference in our approach is that the random variable

$$\frac{x(T)}{\sigma} = \int_0^T P_q^{\frac{1-q}{2}} dz_s = \Omega(T) \quad (82)$$

is not normally distributed, but rather according to the Tsallis distribution of index q Eq(15). The pricing equation Eq(78) can be written as

$$f = \frac{e^{-rT}}{Z(T)} \int_R h \left[S(0) \exp(\sigma \Omega(T) + rT - \frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} (1 - (1-q)\beta(T)\Omega^2(T))) \right] \\ (1 - \beta(T)(1-q)\Omega(T)^2)^{\frac{1}{1-q}} d\Omega_T \quad (83)$$

In the limit $q = 1$, the standard result is recovered.

7 European Call Options

A European call option is such that the option holder has the right to buy the underlying stock S at the strike price K , on the day of expiration T . Depending on the value of $S(T)$, the payoff of such an option is

$$C = \max[S(T) - K, 0] \quad (84)$$

In other words, if $S(T) > K$ then the option will have value (it will be in-the-money). In a more concise notation, the price c of such an option becomes

$$c = E^Q[e^{-rT}C] \quad (85)$$

$$= E^Q[e^{-rT}S(T)]_D - E^Q[e^{rT}K]_D \quad (86)$$

$$= J_1 - J_2 \quad (87)$$

where the subscript D stands for the set $\{S(T) > K\}$. To calculate J_1 and J_2 we shall proceed along the same lines as in the standard case [24]. We have

$$J_2 = e^{-rT}K \left(\int_R \frac{1}{Z(T)} (1 - \beta(T)(1-q)\Omega(T)^2)^{\frac{1}{1-q}} d\Omega_T \right)_D \quad (88)$$

$$= e^{-rT}K \mathbf{P}_Q\{S(T) > K\} \quad (89)$$

where the notation $\mathbf{P}_Q\{S(T) > K\}$ is just a more concise notation for the expression on the line above. \mathbf{P}_Q corresponds to the integral over the Tsallis distribution (which was defined with respect to the measure Q), and the argument $\{S(T) > K\}$ is referring to the fact that we are considering only the set D . We get

$$\begin{aligned}
J_2 &= e^{-rT} K \mathbf{P}_Q\{S(T) > K\} \\
&= e^{-rT} K \mathbf{P}_Q\{S(0) \exp(\sigma\Omega + rT - \frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} (1 - (1-q)\beta(T)\Omega^2)) > K\} \\
&= e^{-rT} K \mathbf{P}_Q\{-\frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} (1 - (1-q)\beta(T)\Omega^2) + \sigma\Omega + rT > \ln \frac{K}{S(0)}\}
\end{aligned} \tag{90}$$

The inequality

$$-\frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} + (1-q) \alpha T^{\frac{2}{3-q}} \beta(T) \frac{\sigma^2}{2} \Omega^2 + \sigma\Omega + rT > \ln \frac{K}{S(0)} \tag{91}$$

is satisfied inbetween the two roots

$$\begin{aligned}
s_{1,2} &= \frac{-1}{\alpha T^{\frac{2}{3-q}} (1-q) \sigma \beta(T)} \\
&\pm \left[\frac{1}{\alpha T^{\frac{4}{3-q}} (1-q)^2 \sigma^2 \beta(T)^2} \right. \\
&\quad \left. - \frac{2}{(1-q) \alpha T^{\frac{2}{3-q}} \sigma^2 \beta(T)} \left(rT + \ln \frac{S(0)}{K} - \frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} \right) \right]^{\frac{1}{2}}
\end{aligned} \tag{92}$$

This is a very different situation from the standard case, where the inequality is linear and the condition $S(T) > K$ is satisfied for all values of the random variable greater than a threshold. In our case, due to the noise induced drift, values of $S(T)$ in the risk-neutral world are not monotonically increasing as a function of the noise. As $q \rightarrow 1$, the larger root goes toward ∞ , recovering the standard case. But as q gets larger, the tails of the noise distribution get larger, as does the noise induced drift which tends to pull the system back. As a result we obtain

$$J_2 = \frac{e^{-rT} K}{Z(T)} \int_{s_1}^{s_2} (1 - (1-q)\beta(T)\Omega^2)^{\frac{1}{1-q}} d\Omega \tag{93}$$

The remaining term J_1 can be determined in a similar fashion. We have

$$J_1 = E^Q[e^{-rT}S(T)]_D \quad (94)$$

This can be written as

$$\begin{aligned} J_1 &= E^Q[e^{-rT}S(T)]_D \\ &= \mathbf{P}_Q[e^{-rT}S(T)]\{S(T) > K\} \\ &= \mathbf{P}_Q[e^{-rT} \exp(S(0) \exp(\sigma\Omega(T) + rT - \frac{\sigma^2}{2}\alpha T^{\frac{2}{3-q}}(1 - (1-q)\beta(T)\Omega^2(T))))] \\ &\quad \{S(T) > K\} \end{aligned} \quad (95)$$

The domain $\{S(T) > K\}$ is the same as that found for J_2 , and is defined as the region between the two roots of Eq(92). We obtain

$$\begin{aligned} J_1 &= \frac{S(0)}{Z(T)} \int_{s_1}^{s_2} \exp(\sigma\Omega - \frac{\sigma^2}{2}\alpha T^{\frac{2}{3-q}} + (1-q)\alpha T^{\frac{2}{3-q}}\beta(T)\frac{\sigma^2}{2}\Omega^2) \\ &\quad (1 - (1-q)\beta(T)\Omega^2)^{\frac{1}{1-q}} d\Omega \end{aligned} \quad (96)$$

It is customary in the standard Black-Scholes case to express the integrals in Eq(96) and Eq(90) in terms of a standardized (0,1) noise process. It is possible to do the same in the generalized case, via the appropriate variable transformation

$$\Omega_N = \Omega(T) \sqrt{\frac{\beta(T)}{\beta_N}} \quad (97)$$

We thus obtain the following expression for a European call option:

$$c = S(0)M_q(d_1, d_2, b(\Omega_N)) - e^{-rT}KN_q(d_1, d_2) \quad (98)$$

where we introduce the notation

$$N_q(d_1, d_2) = \frac{1}{Z_N} \int_{d_1}^{d_2} (1 - (1-q)\beta_N\Omega_N^2)^{\frac{1}{1-q}} d\Omega_N \quad (99)$$

and

$$M_q(d_1, d_2, b(\Omega_N)) = \frac{1}{Z_N} \int_{d_1}^{d_2} \exp(b(\Omega_N))(1 - (1-q)\beta_N\Omega_N^2)^{\frac{1}{1-q}} d\Omega_N \quad (100)$$

with

$$b(\Omega_N) = \sigma \sqrt{\frac{\beta_N}{\beta(T)}} \Omega_N - \frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} (1 - (1-q)\beta_N \Omega_N^2) \quad (101)$$

The limits of the standardized integrals are given as

$$d_{1,2} = \frac{s_{1,2}}{\sigma \sqrt{\beta_N / \beta(T)}} \quad (102)$$

with $s_{1,2}$ as in Eq(92). By choosing β_N as

$$\beta_N = \frac{1}{5 - 3q} \quad (103)$$

the variance of the noise distribution will be normalized to 1 for each value of q . In the limit $q = 1$, the standard Black-Scholes pricing equations are recovered.

8 Martingale Solutions and the Generalized Black-Scholes Differential Equation

We must yet discuss the equivalence of the solution f found via the martingale asset pricing approach, and the solution of the generalized Black-Scholes differential equation (32). We use arguments based on those in [25] for the standard case. The expression for S of Eq(70) can be written for $u \geq t$ as

$$S(u) = S(t) \exp \left(\int_t^u \sigma P_q^{\frac{1-q}{2}} dz_s + \int_t^u \left(r - \frac{\sigma^2}{2} P_q^{1-q} \right) ds \right) \quad (104)$$

This implies that

$$S(T) = S(0) \exp \left(\int_0^T \sigma P_q^{\frac{1-q}{2}} dz_s + \int_0^T \left(r - \frac{\sigma^2}{2} P_q^{1-q} \right) ds \right) \quad (105)$$

can trivially be rewritten as

$$S(T) = S(t) \exp \left(\int_t^T \sigma P_q^{\frac{1-q}{2}} dz_s + \int_t^T \left(r - \frac{\sigma^2}{2} P_q^{1-q} \right) ds \right) \quad (106)$$

$$= XY \quad (107)$$

where

$$X = S(t) \quad (108)$$

$$Y = \exp\left(\int_t^T \sigma P_q^{\frac{1-q}{2}} dz_s + \int_t^T \left(r - \frac{\sigma^2}{2} P_q^{1-q}\right) ds\right) \quad (109)$$

with the important properties that X is measurable with information $I(t)$ available up until time t , and Y is independent of that information.

We then define

$$v(t, X) = E^Q[h(S(T)) | I(t)] \quad (110)$$

$$= E^Q[h(X \exp(\int_t^T \sigma P_q^{\frac{1-q}{2}} dz_s + \int_t^T (r - \frac{\sigma^2}{2} P_q^{1-q}) ds)] \quad (111)$$

where h is an arbitrary function. We now look at the value of this expectation conditioned on information $I(t)$ available up until time t and obtain

$$E^Q[h(S(T)) | I(t)] = E^Q[h(XY) | I(t)] \quad (112)$$

$$= E^Q[h(X) | I(t)] \quad (113)$$

$$= v(t, X) \quad (114)$$

$$= v(t, S(t)) \quad (115)$$

where the independence of Y on $I(t)$ has been used. This is exactly the same result as obtained in the standard case, and it implies that $v(t, S(t)), 0 \leq t \leq T$, is a martingale [25]. We proceed to use Ito's formula to write

$$dv(t, S(t)) = \left[\frac{dv}{dt} + rS\frac{dv}{dS} + \frac{1}{2}\sigma^2 S^2 P_q^{1-q} \frac{d^2v}{dS^2}\right]dt + \sigma S \frac{dv}{dS} P_q^{\frac{1-q}{2}} dz \quad (116)$$

But because v is a martingale, we know that the sum of the dt terms must equal 0. This implies that

$$\frac{dv}{dt} + rS\frac{dv}{dS} + \frac{1}{2}\sigma^2 S^2 P_q^{1-q} \frac{d^2v}{dS^2} = 0 \quad (117)$$

for $0 \leq t \leq T$, which is consistent with the Feynman-Kac theorem (cf [25, 23]), albeit now generalized to the current framework.

Recall that the price of a contingent claim paying $h(S(T))$ can be written as Eq(76) so that

$$f = E^Q[e^{-r(T-t)}C] \quad (118)$$

$$= e^{-r(T-t)}E^Q[h(S(T))] \quad (119)$$

$$= e^{-r(T-t)}v(S, t) \quad (120)$$

implying that

$$v(S, t) = e^{r(T-t)}f \quad (121)$$

Insertion of this form of v into Eq(117) immediately yields our generalized Black-Scholes partial differential equation of Eq(32).

We have thus shown that the option price f obtained by way of transforming the asset price into a martingale and discounting it accordingly (as represented by Eq(76)) in turn implies that the generalized Black-Scholes equation of Eq(32) must be valid. Therefore, equivalent solutions can be found either by solving Eq(32) or Eq(76).

9 Dividends and Futures

We shall now show that the current model can also be generalized in a straightforward way to give the price of options on dividend paying stocks, as well as options on futures contracts of the underlying stock. The futures markets are widely traded, so being able to price these instruments within the current framework could be very useful.

We first look at the case of a dividend paying stock. Following standard arguments [5], in time Δt the portfolio $\Delta\Pi$ (Eq (28)) gains wealth equal to $\Delta\Pi$ as in Eq (29) as well as dividends equal to

$$wS\frac{\partial f}{\partial S}\Delta t \quad (122)$$

where w denotes a continuous dividend yield. The generalized Black-Scholes differential equation thus becomes

$$\frac{df}{dt} + (r - w)S\frac{df}{dS} + \frac{1}{2}\frac{d^2f}{dS^2}\sigma^2S^2P_q^{1-q} = rf \quad (123)$$

In the risk-neutral martingale representation, this is equivalent to taking the discounted expectation of a stock yielding a return $r - w$. For European calls we obtain

$$c = S(0)e^{-wT}M_q(e_1, e_2, b(\Omega_N)) - e^{-rT}KN_q(e_1, e_2) \quad (124)$$

using the same notation as in Eq(99) and Eq(100), but where e_1 and e_2 are solutions to

$$\begin{aligned} e_{1,2} &= \frac{-1}{\alpha T^{\frac{2}{3-q}}(1-q)\sigma\beta(T)} \\ &\pm \left[\frac{1}{\alpha T^{\frac{4}{3-q}}(1-q)^2\sigma^2\beta(T)^2} \right. \\ &\quad \left. - \frac{2}{(1-q)\alpha T^{\frac{2}{3-q}}\sigma^2\beta(T)} \left((r-w)T + \ln \frac{S(0)}{K} - \frac{\sigma^2}{2}\alpha T^{\frac{2}{3-q}} \right) \right]^{\frac{1}{2}} \end{aligned} \quad (125)$$

The evaluation of options on futures is now straightforward, since one argues that a futures contract F is equivalent to a stock paying dividends w exactly equal to the risk-free rate of return r . Therefore, for this case we obtain

$$\frac{df}{dt} \frac{1}{2} \frac{d^2 f}{dF^2} \sigma^2 F^2 P_q^{1-q} = rf \quad (126)$$

The closed form solution for European calls follows as

$$c = e^{-rT}[F(0)M_q(e_1, e_2, b(\Omega_N)) - KN_q(e_1, e_2)] \quad (127)$$

with e_1 and e_2 given as in Eq(125) with $S(0)$ substituted by $F(0)$ and $w = r$. For $q = 1$, this is known as the Black model.

10 Numerical Results and The Greeks

We evaluated European call options using Eq(98), and confirmed these results by numerically solving Eq(32) on a grid under appropriate boundary conditions. It is of particular interest to evaluate call options and see how the option prices and partials change as q moves away from 1, which recovers the Black-Scholes scenario.

Results of such calculations are shown in Figures 3 onward. Figure 3 depicts the call option price as a function of the strike price for the standard

Black-Scholes model ($q = 1$) and our model with $q = 1.5$, where σ is chosen such that the at-the-money prices are equal. The differences between the two pricing models is more apparent in Figure 4. There it is clear that both in-the-money and out-of-the money options are valued higher with $q = 1.5$, except for very deep-in-the-money options which are valued lower. This behaviour can be understood intuitively as follows. The distribution of Ω for $q = 1.5$ has fatter tails than the $q = 1$ model. Consequently, if the stock price gets deep out-of-the-money, then the noise may still produce shocks that can bring the stock back in-the-money again. This results in higher option prices for deep out-of-the-money strikes. Similarly, if the option is deep in-the-money, the noise can produce shocks to the underlying which can bring the price out-of-the-money again. In addition, it can be seen from the expression Eq(70) for $S(T)$, that large shocks will increase the value of the noise-induced drift term which will decrease the probability of realizing higher stock prices. This results in lower option prices for deep-in-the-money strikes. On the other hand, for intermediate values around-the-money, there will be a higher probability to land both in- or out-of-the-money which leads to an increase in the option price, relative to the standard $q = 1$ model.

The resulting volatilities which the standard model must assume in order to match the values obtained for the $q = 1.5$ model, are plotted in Figure 5, for $T = 0.1$ and $T = 0.6$. Clearly, these implied volatilities (shown here for values $\pm 20\%$ around-the-money) form a smile shape, very similar to that which is implied by real market data. The higher volatility $q = 1$ Gaussian models that are successively needed as one moves away-from-the money essentially reflects the fact that the tails of the $q = 1.5$ model would have to be approximated by higher volatility Gaussians, whereas the central part of the $q = 1.5$ noise distribution can be approximated by lower volatility Gaussians.

In Figure 6, the call option price as a function of time to expiration T is plotted, for $q = 1$ and $q = 1.5$. Figure 7 shows the call price as a function of the parameter q for $T = 0.4$. As q increases, the three curves corresponding to strikes in-the-money, at-the-money, and out-of-the-money all behave similarly. However, the behaviour looks different for smaller T , as is seen in Figure 8 where $T = 0.05$. In Figure 9, the call option price as a function of σ is shown, for $q = 1$ and $q = 1.5$. In all of these plots, we use parameters close to those in [26], where one can verify our results for $q = 1$.

Figures 10 onward show the so-called Greeks as a function of the current

stock price. The Greeks are partial derivatives defined as

$$\Delta = \frac{\partial f}{\partial S} \quad (128)$$

$$\theta = -\frac{\partial f}{\partial T} \quad (129)$$

$$\kappa = \frac{\partial f}{\partial \sigma} \quad (130)$$

$$\rho = \frac{\partial f}{\partial r} \quad (131)$$

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 f}{\partial S^2} \quad (132)$$

In accordance, we introduce a new Greek designated by the symbol Upsilon (Υ), to represent the partial with respect to q , namely

$$\Upsilon = \frac{\partial f}{\partial q} \quad (133)$$

11 Empirical Results

The real test of any model is how well it can predict or describe empirical data. When it comes to option pricing, the standard Black-Scholes formula misprices observed market prices in a rather systematic way. In particular, if for example σ is chosen so that the theoretical at-the-money call price matches the market price, then the model will underprice out-of-the money and in-the-money calls. Instead, to obtain theoretical prices which match the observed ones, a different value of σ must be used for each value of the strike. A plot of σ versus the strike K is typically a convex function, dubbed the volatility smile. The smile changes also with the time to expiration of the option, flattening out for larger times. A plot of σ over K and T is known as the volatility surface. The fact that this surface is not constant is an indication that the option values predicted by the Black-Scholes model deviate from empirically observed ones.

To test our model, we shall use a value of $q = 1.4$, which is a good fit to the empirically observed returns distribution of financial data such as the S&P 500. We shall then calibrate σ so that the theoretical at-the-money call matches market data, for a given time to expiration T . We shall then

calculate option prices using that one value of σ across different strikes. The next step is to find the different values of σ which a standard Black-Scholes model would need in order to yield the same prices as our $q = 1.4$ model. That will result in a volatility smile, which we can compare to the empirically observed volatility smile. If our model is a good one of market data, then the smiles produced by the model should closely agree with the observed smiles. Furthermore, this should hold true for many different times to expiration.

We have performed just such an experiment on options on Japanese Yen futures. The market data of call prices is readily available, for example on [27]. Market smiles are backed out using a standard Black model (Eq(127) with $q = 1$), and are plotted in Figures 16 and 17. The relevant values of $F(0)$, r and T are noted in the figure captions. We then used the generalized Black model Eq(127) with $q = 1.4$ to obtain theoretical call prices, using one value of σ for each T , chosen so that the theoretical at-the-money all price matched the market at-the-money call (see Figures 16 and 17). Volatility smiles implied by the $q = 1.4$ model were backed out using a standard Black model ($q=1$), and these are also plotted. One sees a very good agreement between market smiles and those implied from our model, for times to expiration ranging from 17 days to 147 days. We would have included longer times to expiration but there was hardly any volume on those options.

We can also plot the at-the-money implied volatility as a function of time to expiration T . This is known as the volatility term structure, and is shown in Figure 18. Interestingly, the $q = 1.4$ at-the-money volatility parameter σ decreases with T , in roughly the same way as the at-the-money σ increases with T for the standard $q = 1$ model. Note however, that while the entire volatility surface consisting of the variation of σ across strikes as well as across time is needed for the $q = 1$ model to fit empirical data, the $q = 1.4$ volatility surface is captured by the plot of σ versus T of Figure 18, seeing as it is that same value of σ which is used across all strikes.

These results are encouraging. In particular, please note that we did not in any way optimize our choice of $q = 1.4$. It may well be that there is another slightly different value of q which well models empirical smiles as well as produces a volatility which is constant with respect to T . Even so, for a practitioner using this model, a deterministic term structure in volatility can easily be hedged away for portfolio management purposes.

Finally, a few brief words comparing the results of our model to other models which have been introduced to accommodate the volatility smile. For

example the DVF (Deterministic Volatility Function) approach [6], where the smile is a consequence of the volatility being time and state dependent. In principle, this is not entirely different to some ideas in our current approach, but in the DVF model the volatility surface shows extreme variations for shorter times to expiration, and is highly nonstationary in general.

Jump diffusion models [4] are another class of interesting stochastic processes which have been introduced to explain smiles. In such models, the volatility smile is explained by adding discontinuous Poisson jumps to the standard Black-Scholes model of stock price movement. These models are however highly parametrized and difficult to handle numerically.

Stochastic volatility models are yet another approach to modeling the volatility smile. Here the volatility is assumed to follow a stochastic process, which could be a GARCH process or a mean-reverting diffusion process (cf [3]). These models again assume more parameters, do not allow for closed-form solutions, and can be difficult to handle numerically. Combinations of some of the approaches listed here have also been studied, for example [7].

With this said, a clear advantage of the approach which we present in this paper, is that we can work with many of the useful tools of the standard Black-Scholes approach, such as techniques from risk-free asset pricing. Numerically, our generalized PDE is easy to solve, and perhaps above all, we obtain closed form solutions for certain special cases. A final point is that we need much fewer parameters to well-model empirically observed option prices, (just one value of σ across all strikes). Our model seems to work well for options on currencies, bonds and perhaps on single-stock options. Even though it also provided a good fit to options on the S & P 100 index [16], in general the volatility smiles observed in such markets tend to have more of a skew, or a “smirk”. Introducing an asymmetry into the noise distribution may be one way of extending our model to such scenarios.

12 Conclusions

In summary, we have proposed modelling the random noise affecting stock returns as evolving across timescales according to an anomalous Wiener process characterised by a Tsallis distribution of index q . This non-Gaussian noise satisfies a statistical feedback process which ultimately depends on a standard Brownian motion. We conclude that our approach yields a better

description than using standard normally distributed noise, because we obtain processes whose distributions match empirical ones much more closely, while including the standard results as a special case. Based on these novel stochastic processes, a generalized form of the Black-Scholes partial differential equation, a closed form option pricing formula, and many other results of mathematical finance can be derived much as is done in the standard theory.

Results generated for the behaviour of the price of a European call option seem to capture some well-known features of real option prices. For example, relative to the standard Black-Scholes model we find that a $q = 1.5$ model gives a higher value to both in-the-money and out-of-the-money options. This means that option prices are quite well modelled using $q = 1.5$ and just one value of the volatility parameter σ across all strikes. As a result of this, we find that the implied volatilities needed for a standard Black-Scholes model ($q = 1$) to match the $q = 1.5$ model show a smile feature across strikes which qualitatively behaves much like empirical observations.

To get a feel for the Greeks of our model, the dependency of the call price on each variable was calculated and plotted for values $q = 1$ and $q = 1.5$. In addition, we introduce a new Greek Υ to represent the variation of the option price with respect to the parameter q . Option prices and partials do deviate significantly from the standard $q = 1$ case as q increases.

Furthermore, we have implemented numerical pricing routines which can be used both for European and American options. These entail implicitly solving the generalized Black Scholes differential equation Eq (32). Results from both methods agree very well, and were further confirmed by calculations involving monte carlo simulations of the underlying stochastic process for the returns.

We must yet study whether the prices obtained for American options match observed ones. However, based on the initial results obtained for European options we are hopeful that this will be the case. We found that $q = 1.4$ with one value of σ across all strikes matches market prices extremely well, at least for the case of calls on Japanese Yen futures studied here, with times to expiration ranging from 17 to 147 days. In this example, the volatility surface for a standard Black model ($q = 1$) is curved across strikes with a slightly upward trending term structure, while the one found for $q = 1.4$ is flat across strikes with a slightly downward trending term structure. Empirical work is still required to see if better option replication can be achieved, and if arbitrage opportunities can be uncovered that do not

appear when the standard model is used. The pricing of exotic options is another topic open for future study.

In closing, we'd like to point out that this work is a first attempt at developing a theory of option pricing based on a noise process evolving according to a nonlinear Fokker-Planck equation. We have assumed that the parameter q is constant for the evolution of returns across all timescales, but one natural theoretical extension of this work could be to let q be a function of the timescale. Another possible extension to the model would be to include an asymmetry in the underlying noise distribution.

A Appendix

A consequence of Ito stochastic calculus is the noise induced drift which appears whenever a transformation of variables occurs. How does this noise induced drift look, and what are its implications, in the generalized case? In the standard case, there are two common and equivalent starting points for modelling the dynamics of stock returns. In the current framework both of those starting points are also valid, and give identical option pricing results. Here we briefly depict the generalized versions of these two models :

One possible starting point start is as in Eq(22), namely

$$dY = \mu dt + \sigma d\Omega \quad (134)$$

with $Y(t) = \ln S(t + \tau)/S(\tau)$ and $d\Omega$ given by Eq(10) yielding

$$dS = (\mu + \frac{\sigma^2}{2} P_q^{1-q}) S dt + \sigma S d\Omega \quad (135)$$

$$= \tilde{\mu} S dt + \sigma S d\Omega \quad (136)$$

Here, the $\sigma^2/2 P_q^{1-q}$ term is a consequence of Ito's Lemma, and corresponds to the noise induce drift term. Alternatively we could choose to start with

$$dS = \mu S dt + \sigma S d\Omega \quad (137)$$

as a model for the stock price evolution across the timescale t , resulting in

$$dY = (\mu - \frac{\sigma^2}{2} P_q^{1-q}) dt + \sigma d\Omega \quad (138)$$

With Eq(137) as a starting point, the noise induced drift term enters in the equation for $\ln S$. The question now is deciding which model to use, Eq(22) or Eq(137)? We can derive option pricing formulas using either one as a starting point. The good news is that, just as in the standard case (recovered for $q = 1$), the two models yield identical option pricing formulas, because either way, μ or $\tilde{\mu}$ disappears under the equivalent martingale measure.

B Appendix

The formula for $S(T)$ (Eq(62) or Eq(70) contains terms of type

$$\int_0^T P(\Omega(s), s)^{1-q} ds \quad (139)$$

But for each value of s , the distribution of the random variable $\Omega(s)$ follows a Tsallis distribution of the form

$$P_q(\Omega(s), s) = \frac{1}{Z(s)} (1 - (1 - q)\beta(s)\Omega(s)^2)^{\frac{1}{1-q}} \quad (140)$$

Each such distribution can be mapped onto the distribution of a standardized random variable x_N through the variable transformation

$$x_N = \sqrt{\frac{\beta(s)}{\beta_N}} \Omega(s) \quad (141)$$

with distribution

$$P_q(x_N) = \frac{1}{Z_N} (1 - (1 - q)\beta_N x_N^2)^{\frac{1}{1-q}} \quad (142)$$

where the standard relation

$$P_q(x_N) = P_q(\Omega(s), s) \frac{\partial \Omega_s}{\partial x_N} \quad (143)$$

holds. Note that we can in turn map the standardized distribution of x_N onto the distribution of the variable $x(T)$ at the fixed timescale T via the variable transformation

$$\Omega(T) = \sqrt{\frac{\beta_N}{\beta(T)}} x_N \quad (144)$$

This result could have also been achieved directly via the variable transformation

$$\Omega(s) = \sqrt{\frac{\beta(T)}{\beta(s)}} \Omega(T) \quad (145)$$

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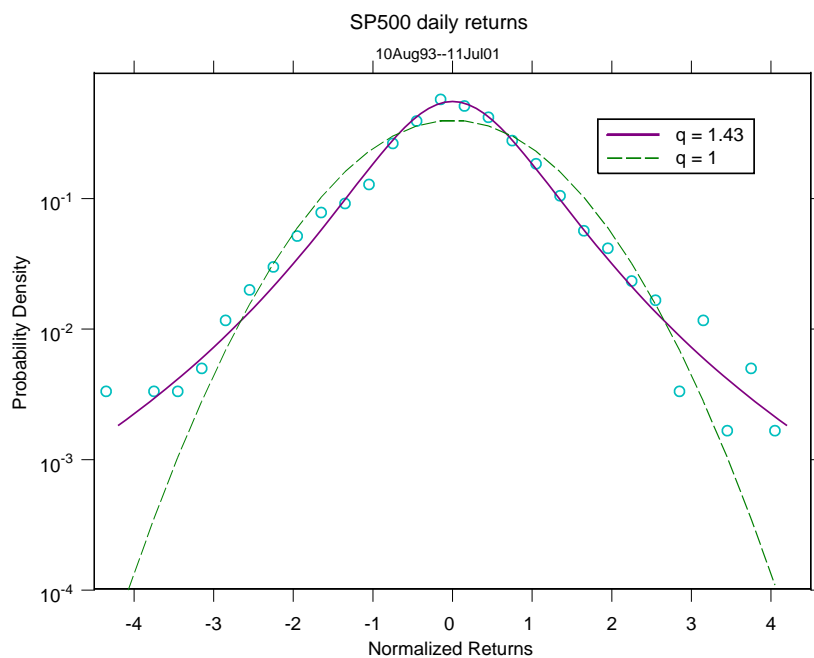


Figure 1: Distributions of log returns, normalized by the sample standard deviation, rising from the demeaned S & P 500, and from a Tsallis distribution of index $q = 1.43$ (solid line). For comparison, the normal distribution is also shown ($q = 1$, dashed line). Figure kindly provided by R. Osorio, to be published [15]

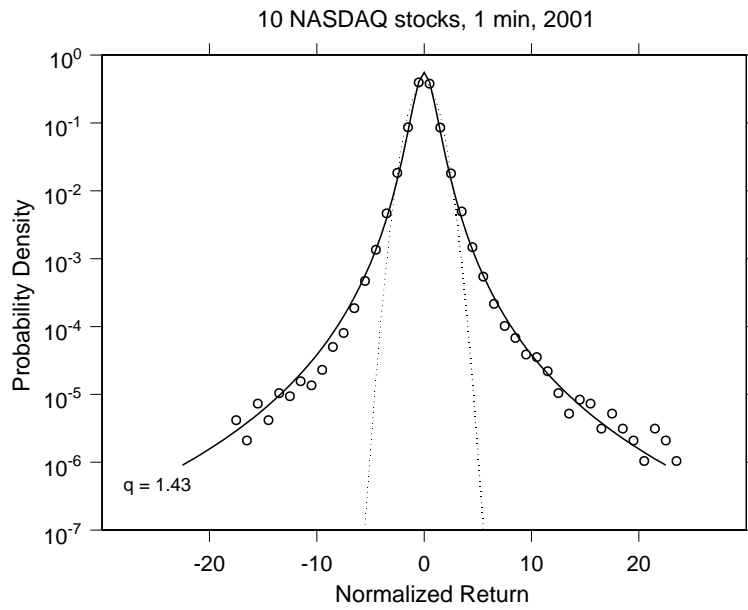


Figure 2: Distributions of log returns for 10 Nasdaq high-volume stocks. Returns are calculated over 1 minute intervals, and are normalized by the sample standard deviation. Also shown is the Tsallis distribution of index $q = 1.43$ (solid line) which provides a good fit to the data. Figure kindly provided by R. Osorio, to be published [15]

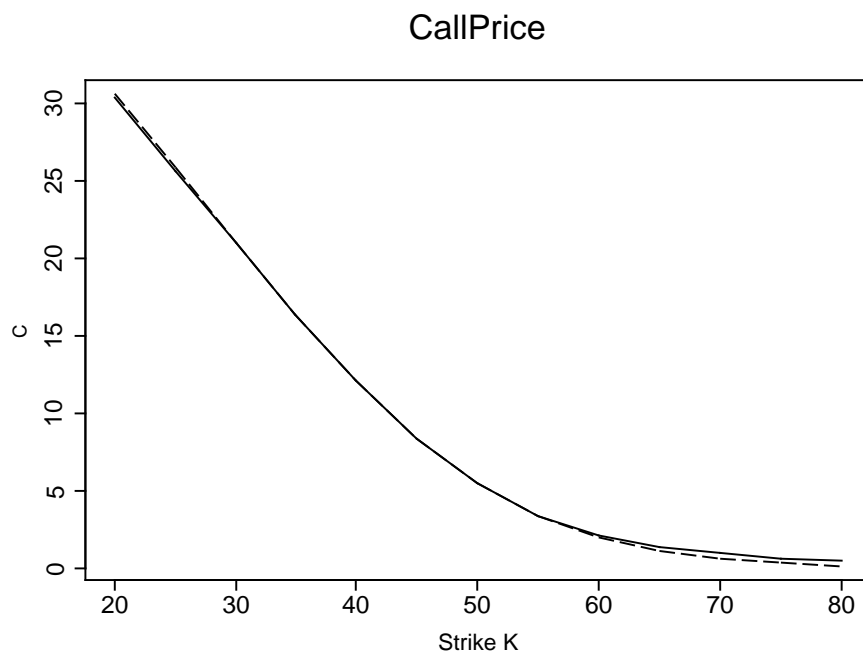


Figure 3: Call option price versus strike price, using $S(0) = 50$, $r = .06$ and $T = .6$, for $q = 1$ (dashed curve) and $q = 1.5$ (solid curve). For each q , σ was chosen so that the at-the-money options are priced equally ($\sigma = .3$ for $q = 1$ and $\sigma = .299$ for $q = 1.5$).

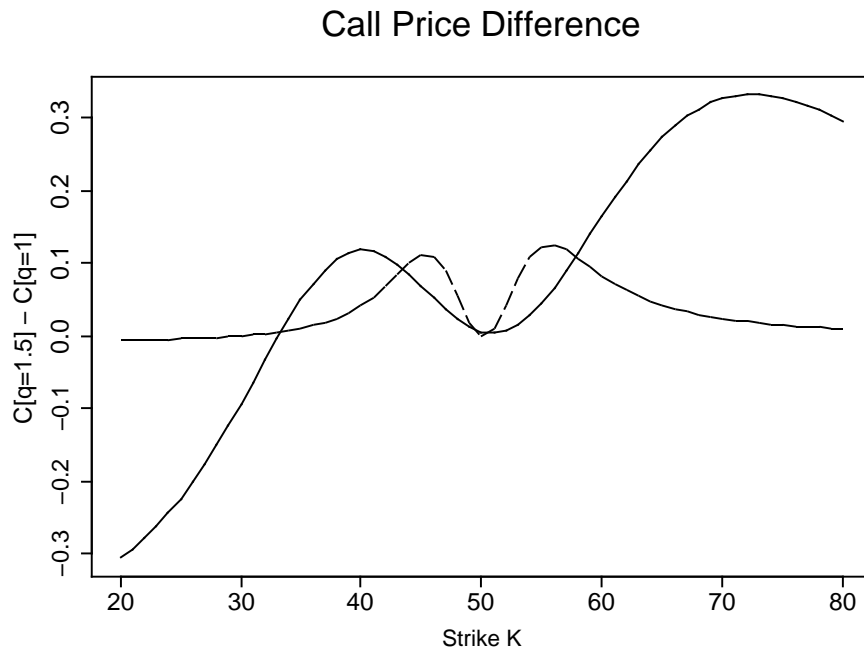


Figure 4: Calibrated so that at-the-money options are priced equally, the difference between the $q = 1.5$ model and the standard Black-Scholes model is shown, for $S(0) = 50\$$ and $r = 0.06$. The solid line corresponds to $T = 0.6$ with $\sigma = .3$ for $q = 1$ and $\sigma = .297$ for $q = 1.5$. The dashed line represents $T = 0.05$ with $\sigma = .3$ for $q = 1$ and $\sigma = .41$ for $q = 1.5$. Times are expressed in years, r and σ are in annual units.

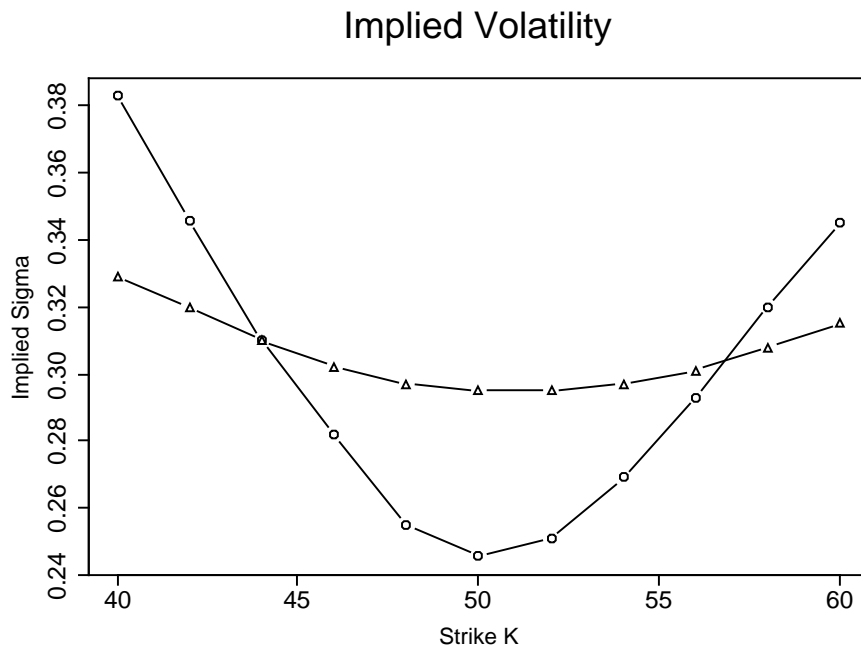


Figure 5: Using the $q = 1.5$ model (here with $\sigma = .3$, $S(0) = 50\$$ and $r = .06$) to generate call option prices, one can back out the volatilities implied by a standard $q = 1$ Black-Scholes model. Circles correspond to $T = 0.4$, while triangles represent $T = 0.1$. These implied volatilities capture features seen in real options data. In particular, the smile is more pronounced for small T .

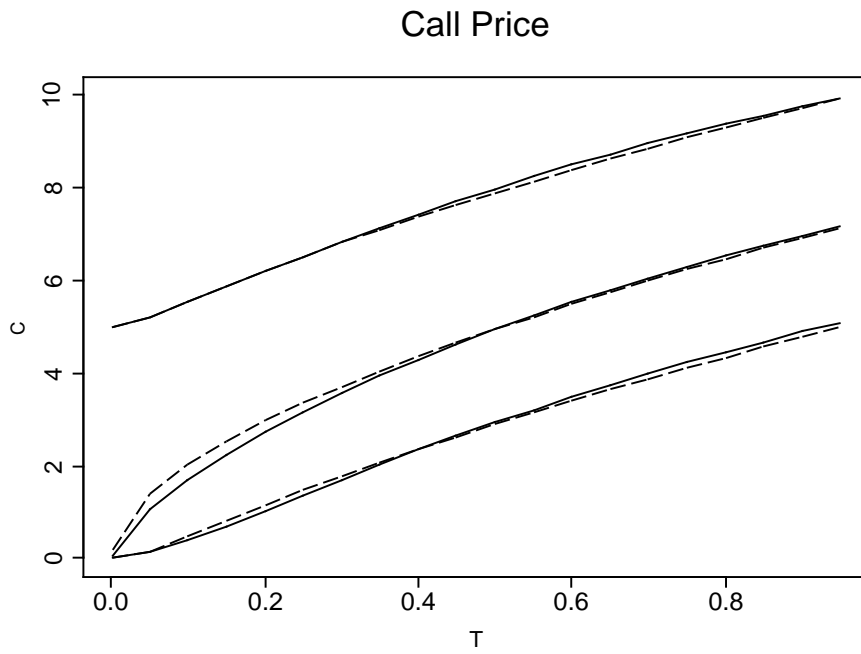


Figure 6: Call option price versus time to expiration, using $S(0) = 50$, $r = .06$, and $\sigma = .3$. Three different strikes were considered: $K = 45$ (in-the-money, top), $K = 50$ (at-the-money, middle) and $K = 55$ (out-of-the-money, bottom). Curves for $q = 1$ (dashed) and $q = 1.5$ (solid) are shown.

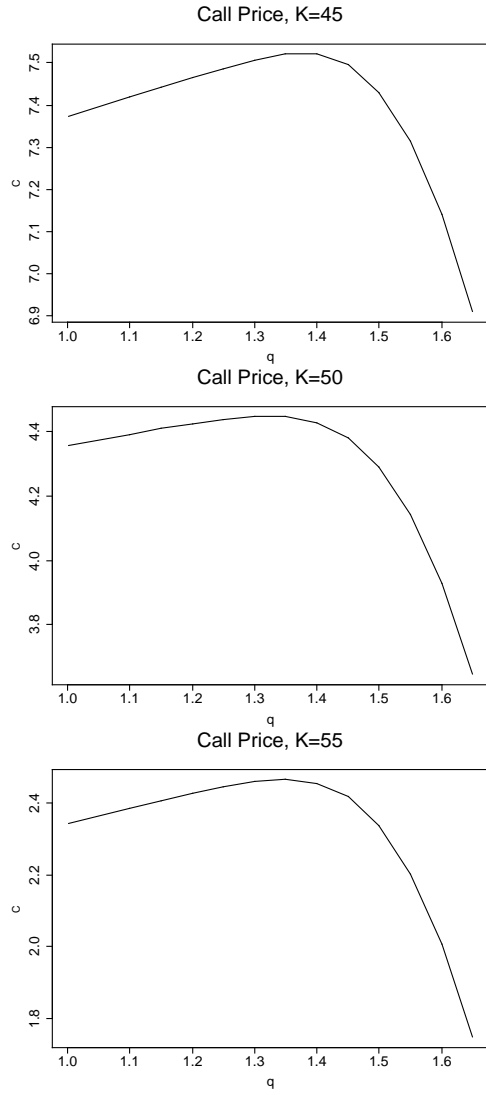


Figure 7: Call option price versus q , using $S(0) = 50$, $r = .06$, and $T = 0.4$. Three different strikes were considered: $K = 45$ (in-the-money, top), $K = 50$ (at-the-money, middle) and $K = 55$ (out-of-the-money, bottom).

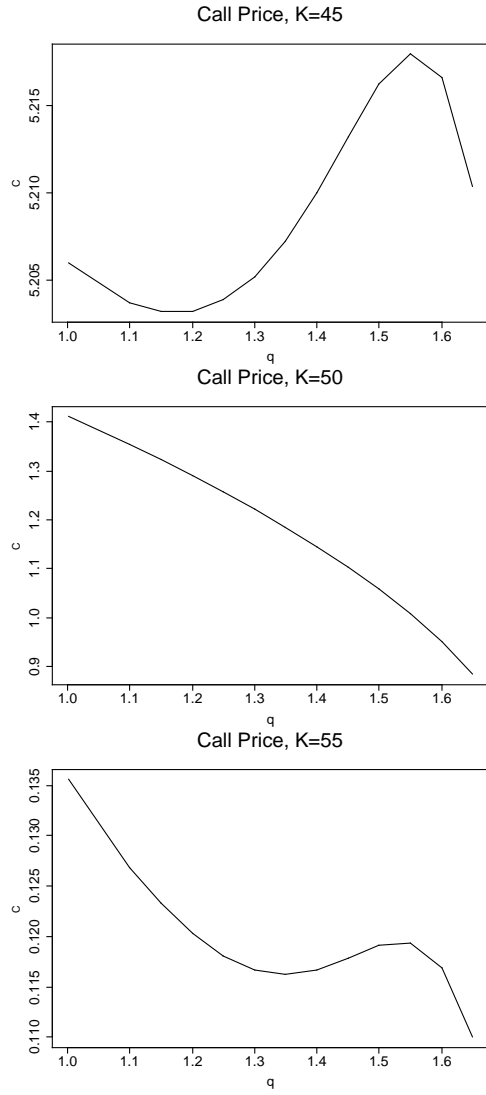


Figure 8: Call option price versus q , using $S(0) = 50$, $r = .06$, and $T = 0.05$. Three different strikes were considered: $K = 45$ (in-the-money, top) $K = 50$ (at-the-money, middle) and $K = 55$ (out-of-the-money, bottom).

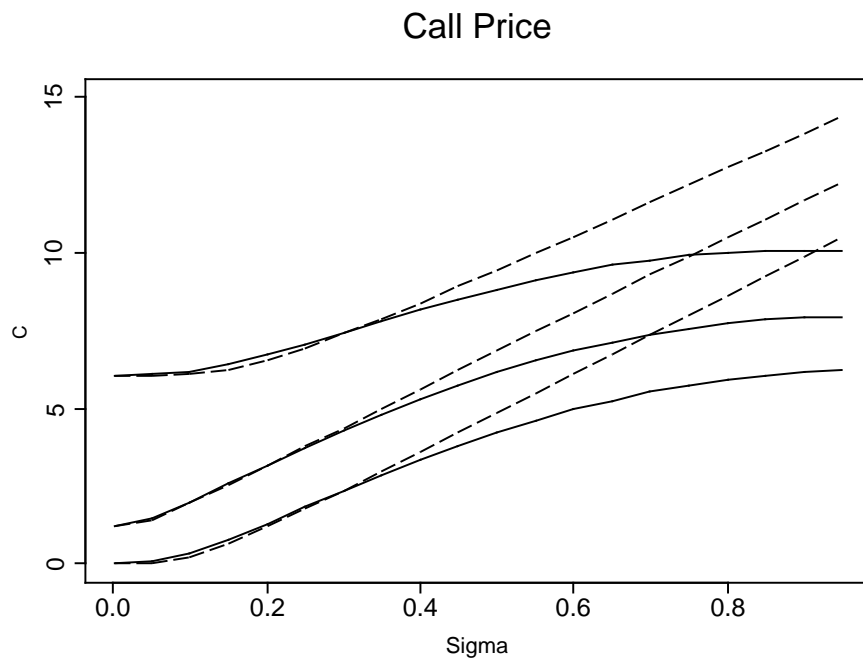


Figure 9: Call option price versus σ , using $S(0) = 50$, $r = .06$, and $T = 0.4$. Three different strikes were considered: $K = 45$ (in-the-money, top), $K = 50$ (at-the-money, middle) and $K = 55$ (out-of-the-money, bottom). Curves for $q = 1$ (dashed) and $q = 1.5$ (solid) are shown.

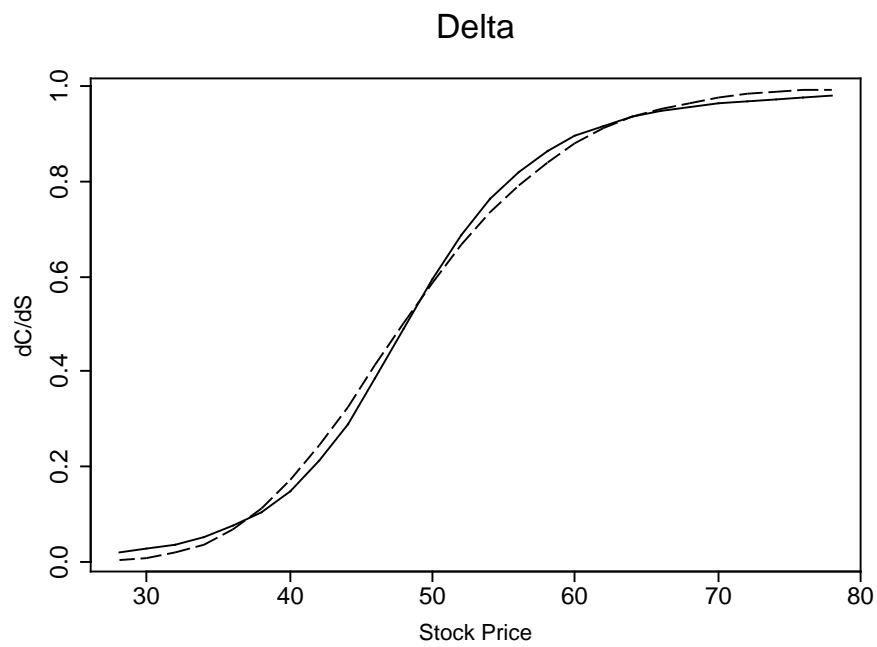


Figure 10: $\Delta = \frac{\partial c}{\partial S}$ as a function of the stock price $S = S(0)$ using $K = 50$, $r = .06$, and $T = 0.4$. Curves for $q = 1$ (dashed) and $q = 1.5$ (solid) are shown.

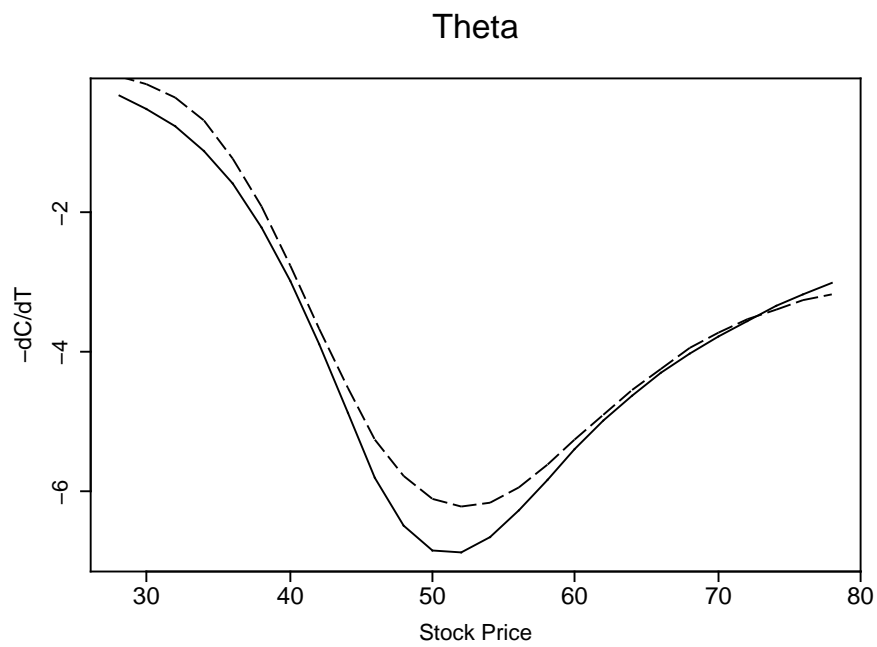


Figure 11: $\Theta = -\frac{\partial c}{\partial T}$ as a function of the stock price $S = S(0)$ using $K = 50$, $r = .06$, and $T = 0.4$. Curves for $q = 1$ (dashed) and $q = 1.5$ (solid) are shown.

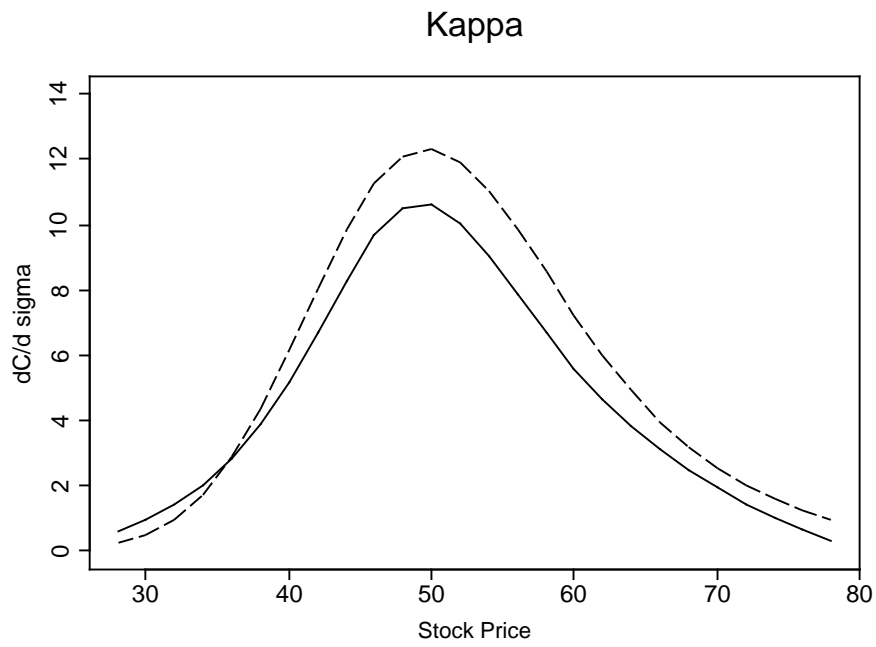


Figure 12: $\kappa = \frac{\partial c}{\partial \sigma}$ as a function of the stock price $S = S(0)$ using $K = 50$, $r = .06$, and $T = 0.4$. Curves for $q = 1$ (dashed) and $q = 1.5$ (solid) are shown.

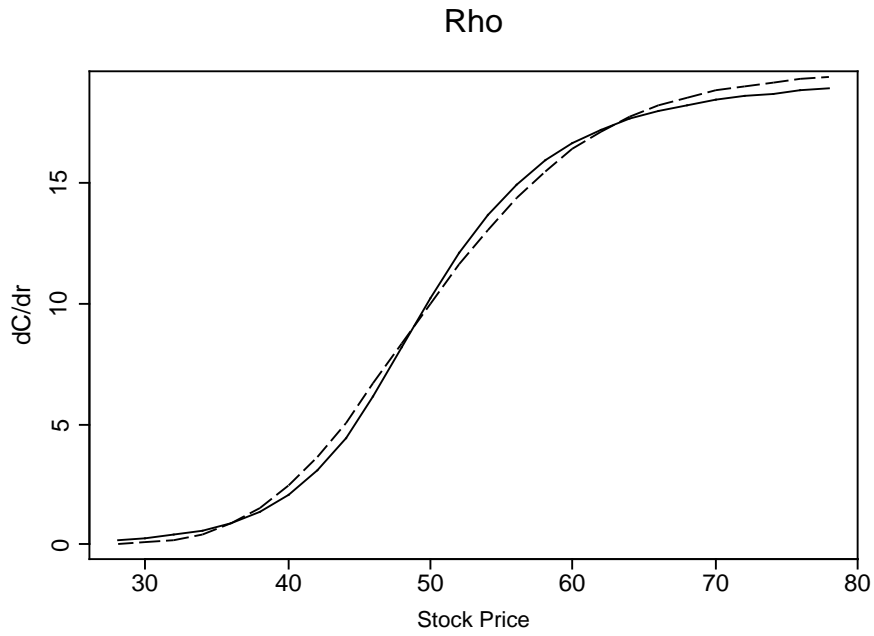


Figure 13: $\rho = \frac{\partial c}{\partial r}$ as a function of the stock price $S = S(0)$ using $K = 50$, $r = 0.06$, and $T = 0.4$. Curves for $q = 1$ (dashed) and $q = 1.5$ (solid) are depicted.

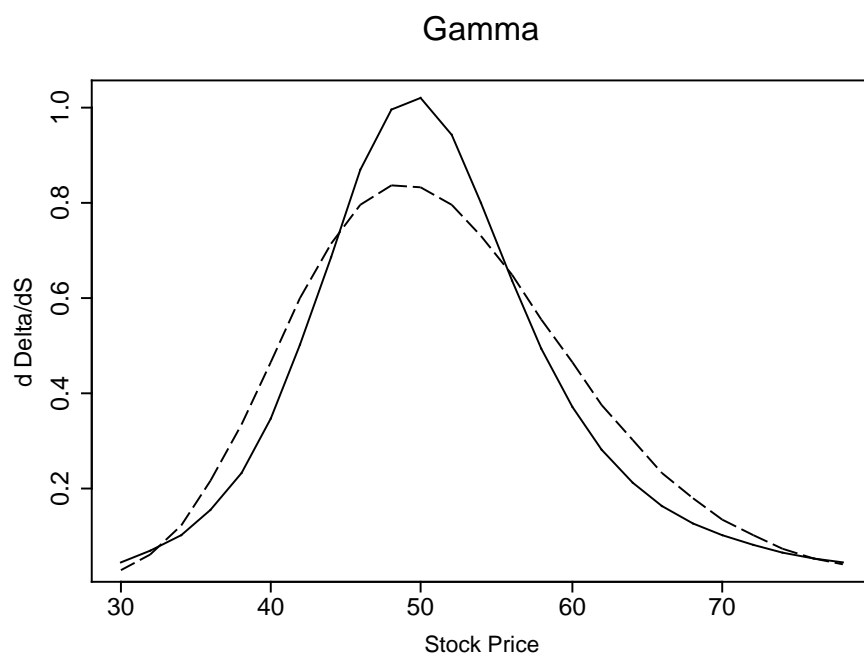


Figure 14: $\Gamma = \frac{\partial \Delta}{\partial S}$ as a function of the stock price $S = S(0)$ using $K = 50$, $r = 0.06$, and $T = 0.4$. Curves for $q = 1$ (dashed) and $q = 1.5$ (solid) are depicted.

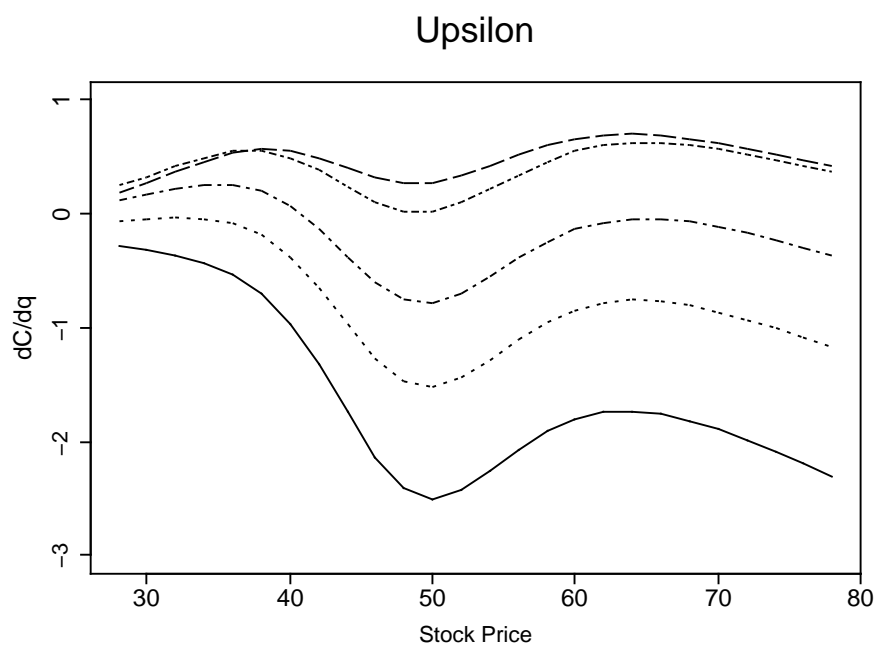


Figure 15: $\Upsilon = \frac{\partial c}{\partial q}$ as a function of the stock price $S = S(0)$ using $K = 50$, $r = .06$, and $T = 0.4$. Curves for $q = 1.1$ (top,dashed) ranging to $q = 1.5$ (solid) are shown. The other curves correspond to $q = 1.3$, $q = 1.4$ and $q = 1.45$ in order of descent.

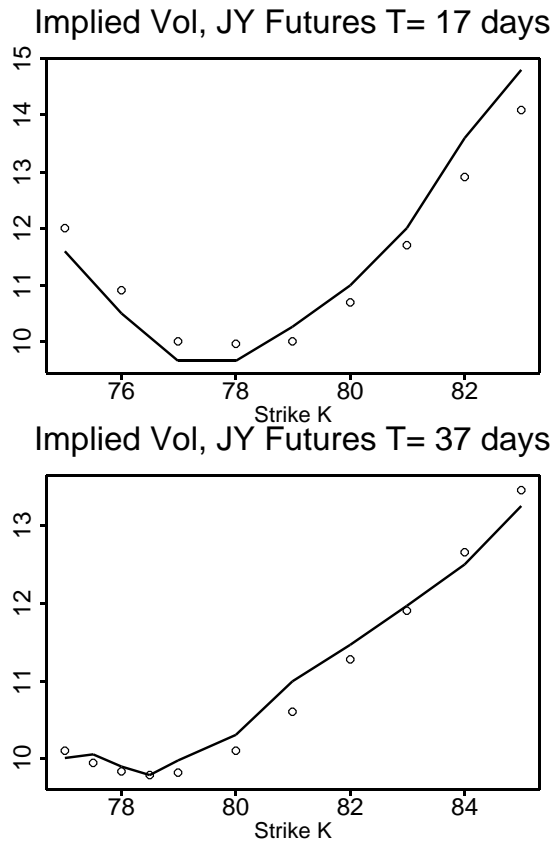


Figure 16: Implied volatilities for call options on JY currency futures, traded on May 16 2002. The following plots will show how smiles implied from our model using $q = 1.4$ match market smiles for times to expiration ranging from about 2 weeks to close to half a year. Option prices were calculated using $q = 1.4$ and just one value of σ for each T (chosen such that the at-the-money option equals the market value). The solid line corresponds to volatilities implied by the market. The symbols correspond to volatilities implied by comparing a standard Black model to ours. $r = .055$ and Top) $F(0) = 78.16$, $\sigma = 12.2\%$ and $T = 17$ days. Bottom) $F(0) = 78.54$, $\sigma = 11.2\%$ and $T = 37$ days.

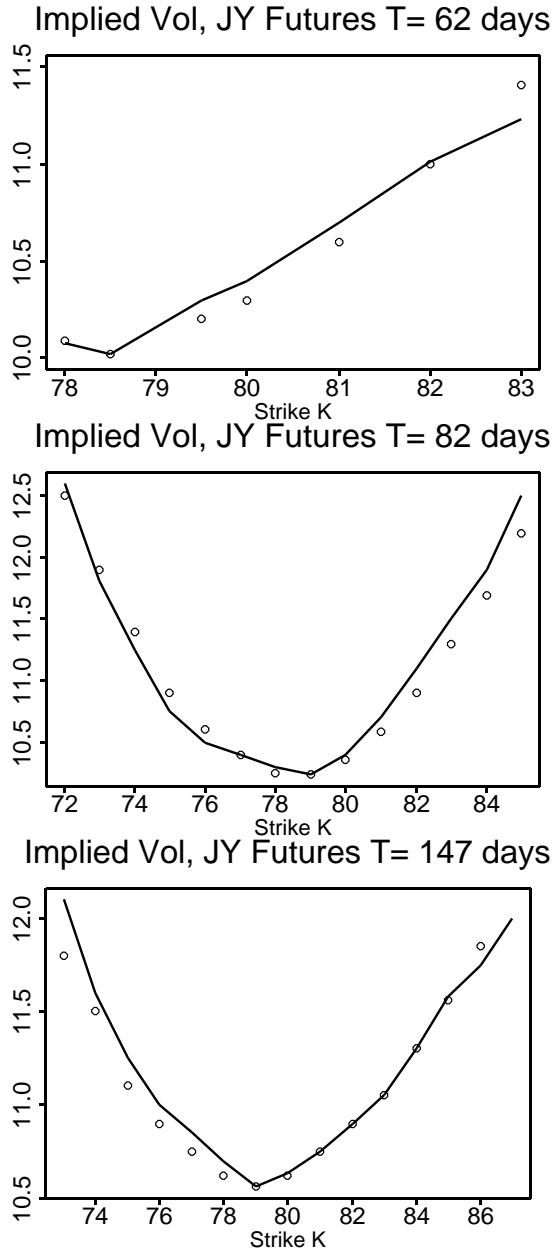


Figure 17: Implied volatilities for call options on JY currency futures, traded on May 16 2002. The solid line corresponds to market implied volatilities. Symbols correspond to volatilities implied by our model with $q = 1.4$, $r = .055$ and Top) $F(0) = 78.54$, $\sigma = 10.8\%$ and $T = 62$ days. Middle) $F(0) = 78.54$, $\sigma = 10.6\%$ and $T = 82$ days. Bottom) $F(0) = 79.01$, $\sigma = 10.2\%$ and $T = 147$ days.

At-The-Money Implied Vols

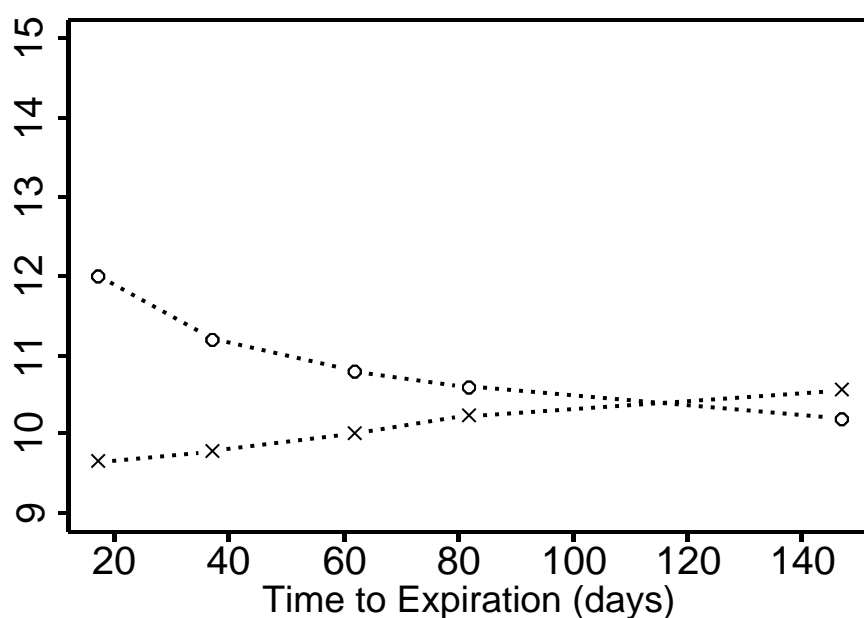


Figure 18: Term structure of the at-the-money volatility in the example of Figure 16 and Figure 17 above. The crosses correspond to at-the-money volatilities of a standard Black model. Circles correspond to at-the-money volatilities of the generalized model with $q = 1.4$. While the volatility surface of the Black model consists of the evolution of smiles with the at-the-money volatility drifting upwards with T , the volatility surface in the $q = 1.4$ model is simply given by a flat surface across strikes, drifting downwards with T .